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Moduli of Bridgeland semistable objects on the projective plane

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Outlook

- X : a smooth projective surface
  $\mathcal{D}(X)$: the bounded derived category of $\mathrm{Coh}(X)$
  Fix $\alpha \in K(X)$ and σ : a Bridgeland stability condition on $\mathcal{D}(X)$

Then we consider the moduli functor $\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma)$ of σ-semistable objects with class α in $K(X)$

- σ : geometric $\Longrightarrow$ $\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma) \cong \mathcal{M}_X(\alpha, \omega)$

$\mathcal{M}_X(\alpha, \omega)$: the moduli functor of ω-semistable coherent sheaves on X,
ω : an ample divisor class in $\mathrm{NS}(X)$

- σ : algebraic $\Longrightarrow$ $\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma) \cong \mathcal{M}_B(\alpha_B, \theta_B)$

$\mathcal{M}_B(\alpha_B, \theta_B)$: the moduli functor of θ_B-semistable modules over a nite dimensional $\mathbb{C}$-algebra B

where $\alpha_B \in K(\mathrm{Mod}\, B)$,　$\mathrm{Mod}\, B$: an abelian category of nitely generated right B-modules

- As an application in the case of $X = \mathbb{P}^2$ we show

$$\mathcal{M}_X(\alpha, \omega) \cong \mathcal{M}_B(\alpha_B, \theta_B)$$

A Bridgeland stability condition σ

σ consists of data $(Z, \mathcal{A})$,

$$Z \colon K(X) \to \mathbb{C}, \quad \mathcal{A} \subset \mathcal{D}(X),$$

Z : a group homomorphism,　$\mathcal{A}$: a full abelian subcategory
These data satisfy some axioms

(For example)
$0 \neq E \in \mathcal{A} \Longrightarrow Z(E) \in \mathbb{R}_{>0} \exp(\sqrt{-1}\pi\phi(E))$
with $0 < \phi(E) \leq 1$

De nition
A nonzero object $E \in \mathcal{A}$: semistable
$\Longleftrightarrow$ $0 \neq A \subset E \Longrightarrow \phi(A) \leq \phi(E)$

Moduli Functors $\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma)$

$\mathcal{M}_{\mathcal{D}(\mathbb{P}^2)}(\alpha, \sigma)$: (scheme/$\mathbb{C}$) $\to$ (sets)

For a $\mathbb{C}$-scheme S,

$\mathcal{M}_{\mathcal{D}(\mathbb{P}^2)}(\alpha, \sigma)(S)$

$= \{E$: a family of σ-semistable objects in $\mathcal{A}$
with class $\alpha \in K(X)\} \subset \mathcal{D}(X \times S)$

Main Theorem

In the case of $X = \mathbb{P}^2$, we nd $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(\mathbb{P}^2)$,

which is both geometric and algebraic.

Main Theorem 0.1 (O)

For any $\alpha \in K(\mathbb{P}^2)$ with $0 < c_1(\alpha) \cdot H \leq r(\alpha)$,
$\Phi_{\mathcal{E}} \colon E \mapsto \mathrm{R\,Hom}_{\mathbb{P}^2}(\oplus E_i, E[1])$ gives the isomorphism

$$\mathcal{M}_{\mathbb{P}^2}(\alpha, H) \cong \mathcal{M}_{B_{\mathcal{E}}}(-\alpha_{B_{\mathcal{E}}}, \theta_Z^{\alpha}),$$

where H is the hyperplane class on $\mathbb{P}^2$ and

(1) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(1), \Omega^1_{\mathbb{P}^2}(3), \mathcal{O}_{\mathbb{P}^2}(2)\}$

or (2) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(3)\}$ *(Le Potier)*

or (3) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(2), \Omega^1_{\mathbb{P}^2}(4), \mathcal{O}_{\mathbb{P}^2}(3)\}$.

$\Leftarrow$

$\Uparrow$

Geometric Stability

Theorem 0.3 (O)

$\alpha \in K(X)$ "normalized", $\mu_\omega(\alpha)$: the slope of α

We take $\beta \uparrow \mu_\omega(\alpha)\omega$ in $\mathrm{NS}(X)_{\mathbb{R}}$. Then

$$\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma_{(\beta,\omega)}) \cong \mathcal{M}_X(\alpha, -\frac{1}{2}K_X, \omega).$$

De nition of $\sigma_{(\beta,\omega)}$ for $\beta, \omega \in \mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$ with ω ample

$\exists \mathcal{A}_{(\beta,\omega)} \subset \mathcal{D}(X)$: tilting of $\mathrm{Coh}(X)$ by $\beta \cdot \omega$

$\exists Z_{(\beta,\omega)} \colon K(X) \to \mathbb{C} \colon E \mapsto -\int \exp(-\beta + \sqrt{-1}\omega)\,\mathrm{ch}(E)$

$\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$ de nes a Bridgeland stability condition on X

Algebraic Stability

If $\exists \mathcal{E} = \{E_0, \cdots, E_n\}$: a full strong exceptional collection on X

Put $B_{\mathcal{E}} = \mathrm{End}_X(\oplus_i E_i)$

Then Bondal showed that $\Phi_{\mathcal{E}} = \mathrm{R\,Hom}_X(\oplus E_i, -)$ gives the equivalence

$$\Phi_{\mathcal{E}} \colon \mathcal{D}(X) \cong \mathcal{D}(\mathrm{Mod}\, B_{\mathcal{E}}).$$

Put $\mathcal{A}_{\mathcal{E}} = \Phi_{\mathcal{E}}^{-1}(\mathrm{Mod}\, B_{\mathcal{E}}) \subset \mathcal{D}(X)$ de ned by pulling back
$\mathrm{Mod}\, B_{\mathcal{E}} \subset \mathcal{D}(\mathrm{Mod}\, B)$ by $\Phi_{\mathcal{E}}$.

Proposition 0.2 (O)

For $\sigma = (Z, \mathcal{A}_{\mathcal{E}})$,

$$\mathcal{M}_{\mathcal{D}(X)}(\alpha, \sigma) \cong \mathcal{M}_{B_{\mathcal{E}}}(\alpha_{B_{\mathcal{E}}}, \theta_Z^{\alpha})$$

where $\alpha_{B_{\mathcal{E}}} = \Phi_{\mathcal{E}}(\alpha) \in K(\mathrm{Mod}\, B_{\mathcal{E}})$ and

θ_Z^{α} is the θ-stability of $B_{\mathcal{E}}$-modules depending on α and Z (and $\mathcal{E}$).

Conclusion and Further Studies

Conclusion

Main Theorem (2) gives another proof of the result by Le Potier (1994).

(Similar results are obtained by Barth in the case of $r(\alpha) = 2$)

$\Longrightarrow$ The Bridgeland stability condition is the useful new concept to study the usual problem.

Further Studies

- Analysis of the wall-crossing phenomena of $\mathcal{M}_{B_{\mathcal{E}}}(-\alpha_{B_{\mathcal{E}}}, \theta_{B_{\mathcal{E}}})$ when $\theta_{B_{\mathcal{E}}}$ varies.

  (the wall-crossing phenomena of $\mathcal{M}_{\mathbb{P}^2}(\alpha, H)$ never occur because $\mathrm{NS}(\mathbb{P}^2)_{\mathbb{R}} = \mathbb{R}H$)

- Our method is applicable for any surface X with a full strong exceptional collection (Generalization).